Technical Note **Fall 2008**

Implementation of Reduced Form Auction for Asymmetric Bidders with Continuous Asymmetric Types

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1 Introduction

For a single object, an auction is a measurable mapping $q : T^N \to [0,1]^N$ satisfying, for all profile $p \in T^N$,

$$\sum_{i=1}^{N} q^i(p) \leq 1, \tag{1}$$

where T is the type space for a generic bidder[1] and N is number of bidders.

Given an auction, q, each bidder i can compute the *interim probability*, $Q^i(t_i)$, that he wins when his type is t_i, by

$$Q^i(t_i) = \int_{T^{N-1}} q^i(t_1, \cdots, t_N) d\mu^{N-1}(t_1, \cdots, \hat{t}_i, \cdots, t_N), \tag{2}$$

where μ is the probability measure on T. Notice that we may drop the index i when the underlying auction q is symmetric. (Maskin and Riley [4] showed that in the i.i.d. case, a seller need only consider a symmetric auction.)

Definition 1 *We say that Q is the* reduced form *of q and that q* implements *Q. Q is said to be* implementable *if there is some symmetric auction q such that* (2) *holds for each $t_i \in T$.*

Matthews [5] conjectured that if a function satisfies the following feasibility MRM Condition (this terminology is given in [3]) for all measurable sets, then it is a reduced form:

MRM Condition: for any given set of types (A), the probability that a winner is of a type from this set ($N \int_A Q d\mu$) must be less than or equal to the probability that there exists a bidder from this set ($1 - \mu(A^c)^N$).

[1] I implicitly assume here that the type spaces identical for all the bidders.

This reduced form representation using only the interim allocation probability has proven to be very useful in reducing the dimension of the problem in practice [1].

Border [2] proved Matthews' conjecture, namely he proved the following proposition:

Proposition 2 *Let $Q : T \to [0,1]$ be measurable. Then Q is implementable by a symmetric auction if and only if for each measurable set of types $A \subset T$, the following inequality holds:*

$$\int_A Q(t)d\mu(t) \leq \frac{1-\mu(A^c)^N}{N}. \quad (3)$$

The necessity part of the result follows very directly from a basic technique of inequality relaxation. The harder part–the sufficiency part of the proof mainly relies on a *separating-hyperplane* argument (the geometric form of Hahn-Banach Theorem).

Border [3] also gives another proof of this characterization of the reduced form (MRM Condition) for even more general case (not necessarily symmetric, Proposition 4) by considering finite type space and taking advantage of the *Theorem of Alternatives* in linear programming, which, at the end of day, can be viewed as a variation of the *separating hyperplane theorem.*

Proposition 3 *The list $\mathbf{P} = (P_1, P_2, \cdots, P_N)$ of functions is the reduced form of a general (not necessarily symmetric) auction $\mathbf{p} = (p_1, p_2, \cdots, p_N)$ if and only if for every subset $A \subset \mathcal{T} := \bigcup_{i=1}^{N} \{i\} \times T_i = \{(i,\tau) : 1 \leq i \leq N, \tau \in T_i\}$ of individual-type pairs, we have*

$$\sum_{(i,\tau)\in A} P_i(\tau)\mu_i^\star(\tau) \leq \mu(\{\mathbf{t} \in \mathbf{T} : \exists (i,\tau) \in A, t_i = \tau\}), \quad (4)$$

where $\mu_i^\star(\cdot)$ is the marginal probability distribution on T_i.

In this note, I will generalize Proposition 3 to the auctions where the bidders's type spaces are *continuous* and *unnecessarily identical.*

2 Continuous Version of Proposition 3

To accomplish this task, I first introduce the notations we are going to use:

There are N unnecessarily symmetric bidders in the market, indexed as $i = 1, 2, \ldots, N$ and the type space of bidder i is denoted as T_i, which I do NOT assume to be finite here. A *profile* of types is simply an element $\mathbf{t} := (t_1, t_2, \cdots, t_N)$ of the product space $T = T_1 \times \cdots \times T_N$. It is a common knowledge (to the sellers and all the bidders) that a particular profile $\mathbf{t}$ is a random realization from the probability distribution μ on T. The pair (T, μ) specifies the *environment*.

I further define the *augmented type space* $\mathcal{T}$ to include the information of identities of the bidders together with the types of each bidder. To be precise, $\mathcal{T}$ is the collection of *identity-type* pairs: [2]

$$\mathcal{T} := \amalg_{i=1}^{N} T_i, \tag{6}$$

where $\amalg_{i=1}^{N}$ is the *disjoint union* operation.

Any subset A of $\mathcal{T}$ now can be expressed as

$$A = \amalg_{i=1}^{N} A_i, \tag{7}$$

where $A_i \subset T_i$ can be $\emptyset$.

An *auction* is defined to be an ordered list of probability assignments $\mathbf{p} = (p_1, p_2, \cdots, p_N)$, where $p_i : T \to [0, 1]$, $i = 1, 2, \ldots, N$ satisfy the following capacity constraint

$$\sum_{i=1}^{N} p_i(\mathbf{t}) \leq 1, \quad \forall \mathbf{t} \in T. \tag{8}$$

What interests bidder i is to compute his or her *interim probability of winning* given his or her own type. In order to express the interim probability, let us introduce more notations. Denote $T^{-i} = \prod_{j \neq i} T_j$ and write $\mathbf{t}^{-i}$ as a generic element in T^{-i}. Hence, $(\tau, \mathbf{t}^{-i}) \in T_i \times T^{-i} = T$ represents a generic element of T.

Let $\mu_i^{\star}(\tau)$ denote the marginal probability distribution on T_i and $\mu_i(\mathbf{t}^{-i}|\tau)$ denote the conditional probability distribution on T^{-i} given that bidder i is of type τ, i.e.

$$\mu_i^{\star}(\tau) := \int_{T^{-i}} \mu(\tau, \mathbf{t}^{-i}) d\mathbf{t}^{-i}, \forall \tau \in T_i; \tag{9}$$

$$\mu_i(\mathbf{t}^{-i}|\tau) := \frac{\mu(\tau, \mathbf{t}^{-i})}{\mu_i^{\star}(\tau)}, \text{ if } \mu_i^{\star}(\tau) > 0, \forall \mathbf{t}^{-i} \in T^{-i}. \tag{10}$$

[2] By definition of *disjoint union*, an equivalent way of defining $\mathcal{T}$ is

$$\mathcal{T} := \cup_{i=1}^{N} \{i\} \times T_i = \{(i, \tau) : 1 \leq i \leq N, \tau \in T_i\}. \tag{5}$$

We say that a subset A of $\mathcal{T}$ is measurable if A_i is measurable as a subset of T_i for all $i = 1, \ldots, N$. We can also define the support of $\mathcal{T}$ under the probability distribution $\mu(\cdot)$ to be

$$\mathcal{T}^\star := \amalg_{i=1}^N T_i^\star, \tag{11}$$

where $T_i^\star := \{\tau \in T_i : \mu_i^\star(\tau) > 0\}$.

For each bidder i and all the type $\tau \in T_i$, we define a function $P_i : T_i \to [0,1]$ as follows:

$$P_i(\tau) := \begin{cases} \int_{T^{-i}} p_i(\tau, \mathbf{t}^{-i}) \mu_i(\mathbf{t}^{-i}|\tau) d\mathbf{t}^{-i}, & \text{if} \quad \tau \in T_i^\star \\ \text{any value} & \text{if} \quad \tau \notin T_i^\star \end{cases} \tag{12}$$

Finally, we are able to formulate the *reduced form* of the auction $\mathbf{p} = (p_1, p_2, \cdots, p_N)$: define a function $\mathbf{P} := \amalg_{i=1}^N P_i : \mathcal{T} \to [0,1]$ as follows

$$\mathbf{P}(\tau) = P_i(\tau), \quad \text{if } \tau \in T_i. \tag{13}$$

Generally speaking, we can define the *disjoint union* of the functions f_i on T_i, denoted as $\mathbf{f} = \amalg_{i=1}^N f_i$, as

$$\mathbf{f}(\tau) = f_i(\tau), \quad \text{if } \tau \in T_i. \tag{14}$$

And for any two functions $\mathbf{f} = \amalg_{i=1}^N f_i$ and $\mathbf{h} = \amalg_{i=1}^N h_i$, we have

$$\mathbf{f} \diamond \mathbf{h} = \amalg_{i=1}^N (f_i \diamond h_i), \tag{15}$$

where $\diamond$ can be any of the operations +, -, $\times$ and $\div$.

Next we define the ambient space we are going to work on. Let $\mathcal{L}_\infty(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star)$ denote the set of all functions on $\mathcal{T}$ that are *essentially bounded* under the norm $\|\mathbf{f}\|_{\mathcal{L}_\infty(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star)} := \sum_{i=1}^N \|f_i\|_{\mathcal{L}_\infty(T_i, \mu_i^\star)}$ (or equivalently[3], $\|\mathbf{f}\|_{\mathcal{L}_\infty(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star)} := \max_{i=1}^N \|f_i\|_{\mathcal{L}_\infty(T_i, \mu_i^\star)}$) for any $\mathbf{f} = \amalg_{i=1}^N f_i$, where $\|f_i\|_{\mathcal{L}_\infty(T_i, \mu_i^\star)}$ is the usual essential bound on the space T_i with probability measure $\mu_i^\star$. For the brevity of notation, we denote $\|\mathbf{f}\|_{\mathcal{L}_\infty(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star)}$ as $\|\mathbf{f}\|_{\mathcal{T}}^\infty$. Similarly, we can define $\mathcal{L}_1(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star)$ equipped with the norm $\|\mathbf{g}\|_{\mathcal{T}}^1 = \sum_{i=1}^N \int_{T_i} |g_i(\tau)| \mu_i^\star(\tau) d\tau$ for any $\mathbf{g} = \amalg_{i=1}^N g_i$. Since $\mathcal{L}_\infty(T_i, \mu_i^\star)$ is the dual of $\mathcal{L}_1(T_i, \mu_i^\star)$ under the usual pairing. It is very easy to see that $\mathcal{L}_\infty(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star)$ is the dual of $\mathcal{L}_1(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star)$ under the pairing

$$\langle \mathbf{f}, \mathbf{g} \rangle = \sum_{i=1}^N \int_{T_i} f_i(\tau) g_i(\tau) \mu^*(\tau) d\tau. \tag{16}$$

[3] I will use these two norms interchangeably in the later.

Now topologize $\mathcal{L}_\infty(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star)$ with its weak*, or $\sigma(\mathcal{L}_\infty, \mathcal{L}_1)$, topology.

For any given $A = \amalg_{i=1}^N A_i \subset \mathcal{T}$, $\chi_A := \amalg_{i=1}^N \chi_{A_i}$ denotes the characteristics function on $\mathcal{T}$ defined as

$$\chi_A(\tau) = \chi_{A_i}(\tau), \quad \text{if } \tau \in T_i, \tag{17}$$

where χ_{A_i} is the usual characteristics function on T_i.

χ_A defines a linear functional on $\mathcal{L}_\infty(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star)$ through (16), namely:

$$\langle \chi_A, \mathbf{P} \rangle := \sum_{i=1}^N \langle \chi_{A_i}, P_i \rangle = \sum_{i=1}^N \int_{A_i} P_i(\tau)\mu_i^\star(\tau)d\tau \tag{18}$$

Since each χ_{A_i} is a linear functional on $\mathcal{L}_\infty(T_i, \mu_i^\star)$, (18) is indeed a linear functional on $\mathcal{L}_\infty(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star)$ under the weak* topology specified above.

With these preparations, we are able to state the continuous version of MRM condition:

Theorem 4 *A measurable function $\mathbf{P} = \amalg_{i=1}^N P_i$ on $\mathcal{T}$ is a reduced form auction (of some auction $\mathbf{p}$)* if and only if *for any measurable*[4] $A \subset \mathcal{T}$,

$$\langle \chi_A, \mathbf{P} \rangle \leq \mu(\{\mathbf{t} \in T : \exists i, \quad s.t. \quad t_i \in A_i\}) \tag{19}$$

Let us start the proof of Theorem 4 by first pointing out what is $\mu(\{\mathbf{t} \in T : \exists i, \quad s.t. \quad t_i \in A_i\})$

Lemma 5 *For any subset $A \subset \mathcal{T}$,*

$$\mu(\{\mathbf{t} \in T : \exists i, \quad s.t. \quad t_i \in A_i\}) = \mu(\cup_{i=1}^N A_i \times T^{-i}), \tag{20}$$

where we notice that $\cup_{i=1}^N A_i \times T^{-i}$ is NOT a disjoint union.

Proof: Obvious. ■

Proof:[Necessity of Theorem 4]

We first notice that

$$\begin{aligned} \langle \chi_{A_i}, P_i \rangle &:= \int_{A_i} P_i(\tau)\mu_i^\star(\tau)d\tau \\ &= \int_{A_i \times T^{-i}} p_i(\mathbf{t})\mu(\mathbf{t})d\mathbf{t} \quad \text{(by (10), (12) and Tonelli's Theorem)} \\ &\leq \int_{\cup_{i=1}^N A_i \times T^{-i}} p_i(\mathbf{t})\mu(\mathbf{t})d\mathbf{t}. \end{aligned} \tag{21}$$

[4] A subset $A = \amalg_{i=1}^N A_i \subset \mathcal{T}$ is said to be *measurable* if A_i is measurable in T_i.

Therefore,

$$\begin{aligned}
\langle \chi_A, \mathbf{P} \rangle &= \sum_{i=1}^{N} \langle \chi_{A_i}, P_i \rangle \\
&\le \sum_{i=1}^{N} \int_{\cup_{i=1}^N A_i \times T^{-i}} p_i(\mathbf{t}) \mu(\mathbf{t}) d\mathbf{t} \quad \text{(by (21))} \\
&\le \int_{\cup_{i=1}^N A_i \times T^{-i}} \mu(\mathbf{t}) d\mathbf{t} \quad \text{(by (8))} \\
&= \mu(\{\mathbf{t} \in T : \exists i, \quad s.t. \quad t_i \in A_i\}) \quad \text{(by (20))}
\end{aligned}$$

■

To prove the reverse direction, we intend to use a infinite-dimension version of *separating hyperplane argument* (Hahn-Banach Theorem). In order to accomplish this, let $\mathfrak{P}$ denote the set of all the reduced form auctions and I will show that $\mathfrak{P}$ is a *compact convex* subset of $\mathcal{L}_\infty(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star)$ under the weak* topology specified previously.

The convexity of $\mathfrak{P}$ is pretty clear: suppose $\mathbf{P}, \tilde{\mathbf{P}} \in \mathfrak{P}$ are the reduced form of the auction p and $\tilde{p}$ respectively; then $\alpha\mathbf{P} + (1-\alpha)\tilde{\mathbf{P}}$ is the reduced form of the auction $\alpha p + (1-\alpha)\tilde{p}$ and hence $\alpha\mathbf{P} + (1-\alpha)\tilde{\mathbf{P}} \in \mathfrak{P}$.

Lemma 6 $\mathfrak{P}$ *is convex and compact w.r.t.* $\sigma(\mathcal{L}_\infty(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star), \mathcal{L}_1(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star))$.

Proof: Convexity is already justified above. Let $\mathfrak{A}$ be the set of all auctions, i.e.

$$\mathfrak{A} := \left\{ \mathbf{p} = (p_1, \cdots, p_N) \in \prod_{i=1}^{N} \mathcal{L}_\infty(T, \mu) : \sum_{i=1}^{N} p_i(\mathbf{t}) \le 1, \forall \mathbf{t} \in T \right\}. \tag{22}$$

Using a similar argument in the proof of Lemma 5.4 of [2], we can show that $\mathfrak{A}$ is $\sigma(\prod_{i=1}^{N} \mathcal{L}_\infty(T, \mu), \prod_{i=1}^{N} \mathcal{L}_1(T, \mu))$ compact.

Now define $\Lambda : \mathfrak{A} \to \mathfrak{P}$ to be the mapping from an auction $\mathbf{p}$ to its corresponding reduced form $\mathbf{P}$. I want to show that Λ is continuous, where $\prod_{i=1}^{N} \mathcal{L}_\infty(T, \mu)$ is equipped with the weak* topology ($\sigma(\prod_{i=1}^{N} \mathcal{L}_\infty(T, \mu), \prod_{i=1}^{N} \mathcal{L}_1(T, \mu))$) and $\mathcal{L}_\infty(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star)$ is equipped with the weak* topology $\sigma(\mathcal{L}_\infty(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star), \mathcal{L}_1(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star))$.

Let $\mathbf{p}^{(k)} \to \mathbf{p}$ in $\sigma(\prod_{i=1}^{N} \mathcal{L}_\infty(T, \mu), \prod_{i=1}^{N} \mathcal{L}_1(T, \mu))$. Let $\mathbf{f} \in \mathcal{L}_1(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star)$, define $\tilde{\mathbf{f}} =$

$(\tilde{f}_1, \cdots, \tilde{f}_N) \in \prod_{i=1}^{N} \mathcal{L}_1(T, \mu)$ by $\tilde{f}_i(\mathbf{t}) = f_i(t_i)$. Then, by definition (12)

$$\langle \mathbf{f}, \Lambda \mathbf{p}^{(k)} \rangle = \sum_{i=1}^{N} \int_T f_i(t_i) p_i^{(k)}(\mathbf{t}) \mu(\mathbf{t}) d\mathbf{t} = \langle \tilde{f}, \mathbf{p}^{(k)} \rangle. \tag{23}$$

Since $\mathbf{p}^{(k)} \to \mathbf{p}$ in $\sigma(\prod_{i=1}^{N} \mathcal{L}_\infty(T, \mu), \prod_{i=1}^{N} \mathcal{L}_1(T, \mu))$, we have $\langle \tilde{f}, \mathbf{p}^{(k)} \rangle$ converges to $\langle \tilde{f}, \mathbf{p} \rangle = \langle \mathbf{f}, \Lambda \mathbf{p} \rangle$. Since $\mathbf{f} \in \mathcal{L}_1(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star)$ is arbitrary, we have $\Lambda \mathbf{p}^{(k)} \to \Lambda \mathbf{p}$ in the weak* topology (i.e. $\sigma(\mathcal{L}_\infty(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star), \mathcal{L}_1(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star))$), proving the continuity of Λ.

Now since $\mathfrak{P} = \Lambda(\mathfrak{A})$ with $\mathfrak{A}$ weak* compact and Λ continuous in the above sense, $\mathfrak{P}$ is compact w.r.t. $\sigma(\mathcal{L}_\infty(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star), \mathcal{L}_1(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star))$.

■

In order to transform the infinite-dimension into a finite-dimension argument, we now introduce the simple functions on $\mathcal{T}$. The simple functions on $\mathcal{T}$ are the linear combination of finitely many characteristics functions on $\mathcal{T}$, namely, any simple function on $\mathcal{T}$ is the *disjoint union* of the simple functions on T_i, i.e. of the following form:

$$\amalg_{i=1}^N \sum_{j=1}^{L_i} \alpha_j^i \chi_{A_i^j}, \tag{24}$$

where $\{A_i^1, \cdots, A_i^{L_i}\}$ are finitely many pairwise disjoint subsets of T_i and $\chi_{A_i^j}$ are the characteristics functions on T_i and hence $\sum_{j=1}^{L_i} \alpha_j^i \chi_{A_i^j}$ is the characteristics function on T_i.

Lemma 7 *The simple functions on $\mathcal{T}$ are dense in $\mathcal{L}_1(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star)$.*

Proof: Let $\mathbf{f} = \amalg f_i \in \mathcal{L}_1(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star)$ be any given function and its norm is, by definition,

$$\|\mathbf{f}\|_{\mathcal{T}}^1 = \sum_{i=1}^{N} \|f_i\|_{\mathcal{L}_1(T_i, \mu_i^\star)}. \tag{25}$$

And clearly, we have $f_i \in \mathcal{L}_1(T_i, \mu_i^\star)$ for all $i = 1, 2, \ldots$, N. Since the simple functions on T_i are dense in $\mathcal{L}_1(T_i, \mu_i^\star)$. Hence, for any $\epsilon > 0$, there exists a simple function $\sum_{j=1}^{L_i} \alpha_j^i \chi_{A_i^j}$ on T_i for each $i = 1, 2, \ldots, N$ such that $\|f_i - \sum_{j=1}^{L_i} \alpha_j^i \chi_{A_i^j}\|_{\mathcal{L}_1(T_i, \mu_i^\star)} \leq \epsilon / N$. Then, by definition, we have

$$\|\mathbf{f} - \amalg_{i=1}^N \sum_{j=1}^{L_i} \alpha_j^i \chi_{A_i^j}\|_{\mathcal{T}}^1 = \sum_{i=1}^{N} \|f_i - \sum_{j=1}^{L_i} \alpha_j^i \chi_{A_i^j}\|_{\mathcal{L}_1(T_i, \mu_i^\star)} \leq \epsilon. \tag{26}$$

Therefore, we show that the simple functions on $\mathcal{T}$ are dense in $\mathcal{L}_1(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star)$. ■

Lemma 8 *Let* $\bar{\mathbf{P}} = \amalg_{i=1}^N \bar{P}_i \in \mathcal{L}_\infty(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star)$ *be such that* $\bar{P}_i : T_i \to [0,1]$ *(the ball of radius* N *in* $\mathcal{L}_\infty(\mathcal{T}, \amalg_{i=1}^N \mu_i^\star)$*). And suppose the simple function* $\amalg_{i=1}^N \sum_{j=1}^{L_i} \alpha_j^i \chi_{A_i^j}$ *separates* $\bar{\mathbf{P}}$ *from* $\mathfrak{P}$*, i.e. for all* $\mathbf{P} \in \mathfrak{P}$*,*

$$\langle \amalg_{i=1}^N \sum_{j=1}^{L_i} \alpha_j^i \chi_{A_i^j}, \bar{\mathbf{P}} \rangle > \langle \amalg_{i=1}^N \sum_{j=1}^{L_i} \alpha_j^i \chi_{A_i^j}, \mathbf{P} \rangle. \tag{27}$$

Then for some measurable set $A = \amalg_{i=1}^N A_i \in \mathcal{T}$*,*

$$\langle \chi_A, \bar{\mathbf{P}} \rangle > \mu(\cup_{i=1}^N A_i \times T^{-i}) \tag{28}$$

Proof: By definition, we may assume that all the subsets $\{A_i^j\}$ are pairwise disjoint. Now, rearrange the order of $\{A_i^j\}$ to be

$$A_{i_1}^{j_1}, A_{i_2}^{j_2}, \cdots, A_{i_L}^{j_L},$$

where $L = L_1 + L_2 + \cdots + L_N$ so that the corresponding constant coefficients are sorted in descending order[5]:

$$\alpha_{i_1}^{j_1} > \alpha_{i_2}^{j_2} > \cdots > \alpha_{i_K}^{j_K} > 0 \geq \alpha_{i_{K+1}}^{j_{K+1}} > \cdots > \alpha_{i_L}^{j_L}. \tag{29}$$

Denote the sequence of subsets $\left\{A_{i_1}^{j_1}, A_{i_2}^{j_2}, \cdots, A_{i_K}^{j_K}\right\}$ as $\mathscr{A}$ and define an auction $\mathbf{q}^{\mathscr{A}}$ *generated by the sequence of subsets* $\mathscr{A}$ as follows[6]:

$$\begin{aligned}
\forall \mathbf{t} \in A_{i_1}^{j_1} \times T^{-i_1}, &\qquad q_{i_1}^{\mathscr{A}}(\mathbf{t}) = 1 \text{ and } q_i^{\mathscr{A}}(\mathbf{t}) = 0 \text{ for all } i \neq i_1;\\
\forall \mathbf{t} \in A_{i_2}^{j_2} \times T^{-i_2} \setminus A_{i_1}^{j_1} \times T^{-i_1}, &\qquad q_{i_2}^{\mathscr{A}}(\mathbf{t}) = 1 \text{ and } q_i^{\mathscr{A}}(\mathbf{t}) = 0 \text{ for all } i \neq i_2;\\
\cdots &\\
\forall \mathbf{t} \in A_{i_K}^{j_K} \times T^{-i_K} \setminus \cup_{l=1}^{K-1} A_{i_l}^{j_l} \times T^{-i_l}, &\qquad q_{i_K}^{\mathscr{A}}(\mathbf{t}) = 1 \text{ and } q_i^{\mathscr{A}}(\mathbf{t}) = 0 \text{ for all } i \neq i_K;\\
\forall \mathbf{t} \notin \cup_{l=1}^{K} A_{i_l}^{j_l} \times T^{-i_l}, &\qquad q_i^{\mathscr{A}}(\mathbf{t}) = 0 \text{ for all } i = 1, 2, \ldots, K.
\end{aligned}$$

Notice that the auction defined above depends on the order of the subsets. Even if the subsets are the same, the generated auctions would be totally different if the orders of these subsets are different. Here is an example of the auctions generated by $\mathscr{A} = \{A_1^2, A_2^1, A_1^3, A_1^1, A_2^2\}$ for the case $N = 2$ (see Figure 1).

For comparison, let us also give the auction generated by $\mathscr{A} = \{A_1^3, A_2^2, A_1^2, A_2^1, A_1^1, \}$ for the case $N = 2$ (see Figure 2).

Let $\mathbf{Q}^{\mathscr{A}} = \amalg_{i=1}^N Q_i^{\mathscr{A}} = \Lambda \mathbf{q}^{\mathscr{A}}$, i.e. $\mathbf{Q}^{\mathscr{A}}$ is the reduced form of the auction $\mathbf{q}^{\mathscr{A}}$.

Next, we recursively define K subsets of $\mathcal{T}$: $A(1), A(2), \cdots, A(K)$ as follows:

[5] Let $\mathbf{P} = \mathbf{0}$ in inequality (27) and the nonnegativity of $\bar{\mathbf{P}}$ implies the existence of positive α_i^j.

[6] This particular auction plays the role of *hierarchical auction* proposed in [2].

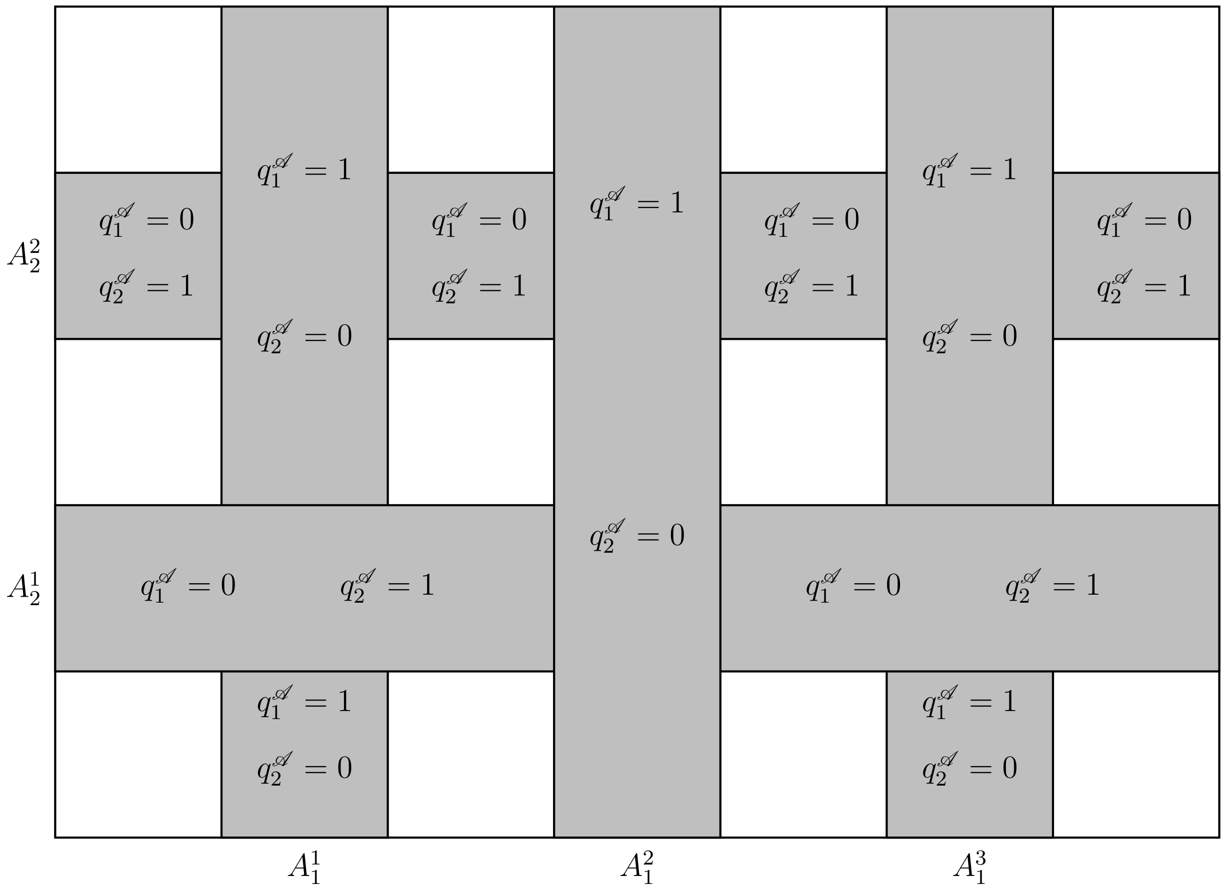

Figure 1: Auction generated by $\mathscr{A} = \{A_1^2, A_2^1, A_1^3, A_1^1, A_2^2\}$ when $N = 2$.

- $A(1) := \amalg_{i=1}^N A_i(1)$, where $A_i(1) \subset T_i$ is given by

$$A_i(1) := \begin{cases} A_{i_1}^{j_1}, & \text{if } i = i_1, \\ \emptyset, & \text{otherwise;} \end{cases}$$

- $A(k) := \amalg_{i=1}^N A_i(k)$, where $A_i(k) \subset T_i$ is given by

$$A_i(k) := \begin{cases} A_i(k-1) \cup A_{i_k}^{j_k}, & \text{if } i = i_k, \\ A_i(k-1), & \text{otherwise.} \end{cases}$$

Claim

$$\langle \chi_{A(k)}, \mathbf{Q}^{\mathscr{A}} \rangle = \mu \left(\bigcup_{l=1}^{k} A_{i_l}^{j_l} \times T^{-i_l} \right) = \mu \left(\bigcup_{i=1}^{N} A_i(k) \times T^{-i} \right) \tag{30}$$

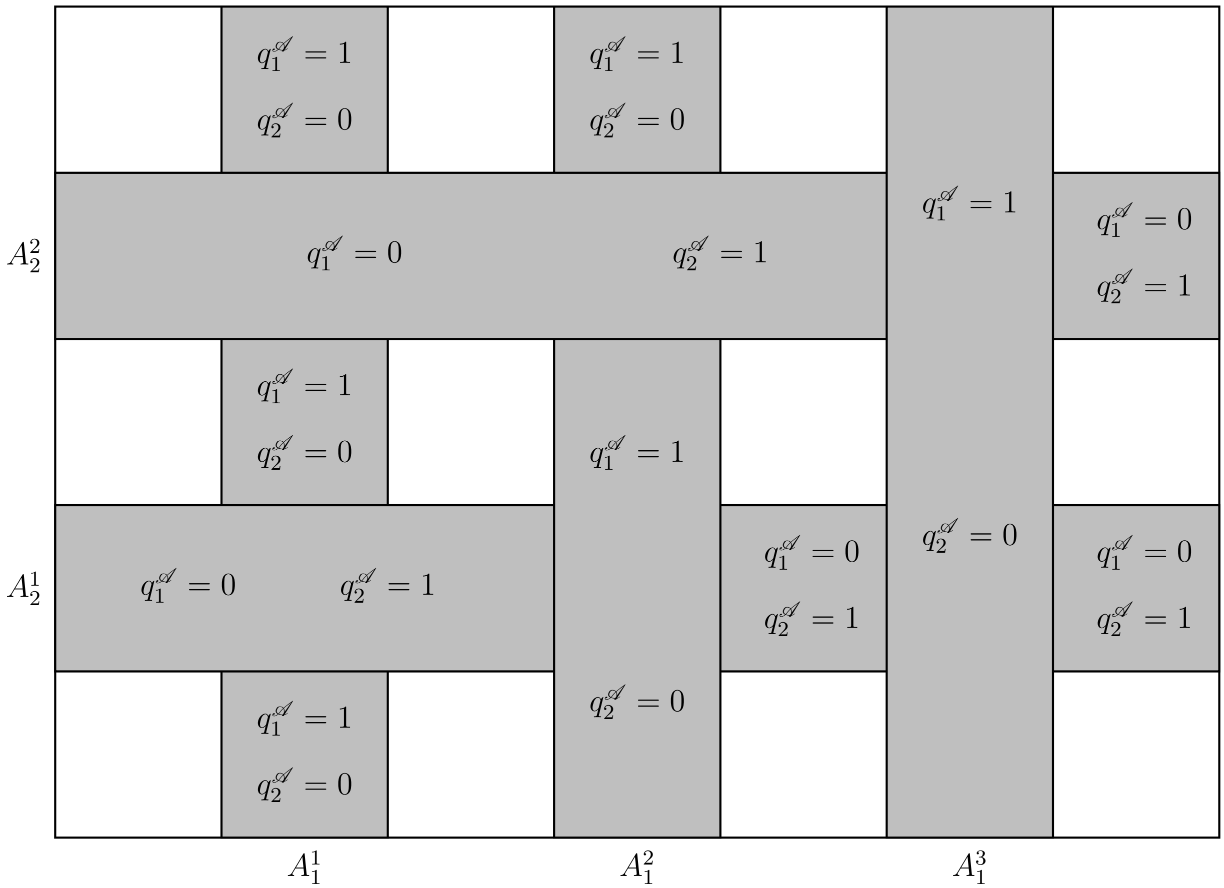

Figure 2: Auction generated by $\mathscr{A} = \{A_1^3, A_2^2, A_1^2, A_2^1, A_1^1,\}$ when $N = 2$.

for $k = 1, 2, \ldots, K$.

We now show this claim by induction on k:

- As defined in (18), we have

$$\begin{aligned}
\langle \chi_{A(1)}, \mathbf{Q}^{\mathscr{A}} \rangle &= \sum_{i=1}^{N} \int_{A_i(1)\times T^{-i}} q_i^{\mathscr{A}}(\mathbf{t})\mu(\mathbf{t})d\mathbf{t} \\
&= \int_{A_{i_1}^{j_1}\times T^{-i_1}} q_{i_1}^{\mathscr{A}}(\mathbf{t})\mu(\mathbf{t})d\mathbf{t} \\
&= \int_{A_{i_1}^{j_1}\times T^{-i_1}} \mu(\mathbf{t})d\mathbf{t} \\
&= \mu(A_{i_1}^{j_1} \times T^{-i_1}),
\end{aligned}$$

  which proves the result for $k = 1$.

- <u>Induction Hypothesis:</u> $\langle \chi_{A(k-1)}, \mathbf{Q}^{\mathscr{A}} \rangle = \mu\left(\bigcup_{l=1}^{k-1} A_{i_l}^{j_l} \times T^{-i_l}\right)$.

- Now we calculate

$$\begin{aligned}
\langle \chi_{A(k)}, \mathbf{Q}^{\mathscr{A}} \rangle &= \sum_{i=1}^{N} \int_{A_i(k)\times T^{-i}} q_i^{\mathscr{A}}(\mathbf{t})\mu(\mathbf{t})d\mathbf{t} \\
&= \sum_{\substack{i=1 \\ i\neq i_k}}^{N} \int_{A_i(k-1)\times T^{-i}} q_i^{\mathscr{A}}(\mathbf{t})\mu(\mathbf{t})d\mathbf{t} + \int_{A_{i_k}(k-1)\cup A_{i_k}^{j_k}\times T^{-i_k}} q_{i_k}^{\mathscr{A}}(\mathbf{t})\mu(\mathbf{t})d\mathbf{t} \\
&= \underbrace{\sum_{i=1}^{N} \int_{A_i(k-1)\times T^{-i}} q_i^{\mathscr{A}}(\mathbf{t})\mu(\mathbf{t})d\mathbf{t}}_{\mu\left(\bigcup_{l=1}^{k-1} A_{i_l}^{j_l}\times T^{-i_l}\right)} \text{ (by Induction Hypothesis)} \\
&\quad + \underbrace{\int_{A_{i_k}^{j_k}\times T^{-i_k}} q_{i_k}^{\mathscr{A}}(\mathbf{t})\mu(\mathbf{t})d\mathbf{t}}_{\mu\left(A_{i_k}^{j_k}\times T^{-i_k}\setminus\bigcup_{l=1}^{k-1} A_{i_l}^{j_l}\times T^{-i_l}\right)} \text{ (by the definition of } \mathbf{q}^{\mathscr{A}}) \\
&= \mu\left(\bigcup_{l=1}^{k} A_{i_l}^{j_l} \times T^{-i_l}\right),
\end{aligned}$$

  which completes the proof of the Claim.

With all the above preparation, we are now ready to prove Lemma 8. Suppose there is a subset $A(k) = \amalg_{i=1}^{N} A_i(k) \in \mathcal{T}$ for some $k = 1, 2, \ldots, K-1$ such that $\langle \chi_{A(k)}, \bar{\mathbf{P}} - \mathbf{Q}^{\mathscr{A}} \rangle > 0$, then we prove (28), because the above claim says:

$$\langle \chi_{A(k)}, \mathbf{Q}^{\mathscr{A}} \rangle = \mu\left(\bigcup_{i=1}^{N} A_i(k) \times T^{-i}\right).$$

So suppose the opposite:

$$\langle \chi_{A(k)}, \bar{\mathbf{P}} - \mathbf{Q}^{\mathscr{A}} \rangle \leq 0, \tag{31}$$

for all $k = 1, 2, \ldots, K-1$. Denote $Q_i^{\mathscr{A}}(j) := \langle \chi_{A_i^j}, Q_i^{\mathscr{A}} \rangle$ and $\bar{P}_i(j) := \langle \chi_{A_i^j}, \bar{P}_i \rangle$. Using this notation and noticing that all the subsets $A_{i_\iota}^{j_\iota}$ are pairwise disjoint no matter whether they belong to the same T_i or not, we may rewrite the above (31) as

$$\sum_{\iota=1}^{k} \left(Q_{i_\iota}^{\mathscr{A}}(j_\iota) - \bar{P}_{i_\iota}(j_\iota) \right) \geq 0, \tag{32}$$

for all $k = 1, 2, \ldots, K-1$. And also, (27) implies

$$\sum_{\iota=1}^{L} \alpha_{i_\iota}^{j_\iota} \left(\bar{P}_{i_\iota}(j_\iota) - Q_{i_\iota}^{\mathscr{A}}(j_\iota) \right) > 0. \tag{33}$$

Taking the $\iota = 1$ term to the right-hand side and dividing by $\alpha_{i_1}^{j_1} > 0$ yields

$$\sum_{\iota=2}^{L} \frac{\alpha_{i_\iota}^{j_\iota}}{\alpha_{i_1}^{j_1}} \left(\bar{P}_{i_\iota}(j_\iota) - Q_{i_\iota}^{\mathscr{A}}(j_\iota) \right) > \left(Q_{i_1}^{\mathscr{A}}(j_1) - \bar{P}_{i_1}(j_1) \right) \geq 0, \tag{34}$$

where the second inequality follows from (32) for $k = 1$. (29), on the other hand implies that $\alpha_{i_1}^{j_1} / \alpha_{i_1}^{j_2} > 1$. Hence, we may multiply $\alpha_{i_1}^{j_1} / \alpha_{i_1}^{j_2}$ on both side of the first inequality of (34) and then take the $\iota = 2$ term to the right-hand side, yielding

$$\sum_{\iota=3}^{L} \frac{\alpha_{i_\iota}^{j_\iota}}{\alpha_{i_2}^{j_2}} \left(\bar{P}_{i_\iota}(j_\iota) - Q_{i_\iota}^{\mathscr{A}}(j_\iota) \right) > \left(Q_{i_1}^{\mathscr{A}}(j_1) - \bar{P}_{i_1}(j_1) \right) + \left(Q_{i_2}^{\mathscr{A}}(j_2) - \bar{P}_{i_2}(j_2) \right) \geq 0, \tag{35}$$

where the second inequality follows from (32) for $k = 2$. Repeating this process will lead to

$$\sum_{\iota=K+1}^{L} \frac{\alpha_{i_\iota}^{j_\iota}}{\alpha_{i_K}^{j_K}} \left(\bar{P}_{i_\iota}(j_\iota) - Q_{i_\iota}^{\mathscr{A}}(j_\iota) \right) > \sum_{\iota=1}^{K} \left(Q_{i_\iota}^{\mathscr{A}}(j_\iota) - \bar{P}_{i_\iota}(j_\iota) \right). \tag{36}$$

Now we claim that $Q_{i_\iota}^{\mathscr{A}}(j_\iota) = 0$ for all $\iota > K$. By definition,

$$\begin{aligned} Q_{i_\iota}^{\mathscr{A}}(j_\iota) &= \langle \chi_{A_{i_\iota}^{j_\iota}}, Q_{i_\iota}^{\mathscr{A}} \rangle \\ &= \int_{A_{i_\iota}^{j_\iota} \times T^{-i_\iota}} q_{i_\iota}^{\mathscr{A}}(\mathbf{t}) \mu(\mathbf{t}) d\mathbf{t}. \end{aligned}$$

Now notice that for all $\kappa = 1, 2, \ldots, K$ such that $i_\kappa = i_\iota$, all $A_{i_\kappa}^{j_\kappa} \times T^{-i_\kappa}$ and $A_{i_\iota}^{j_\iota} \times T^{-i_\iota}$ are pairwise disjoint. Hence, $A_{i_\iota}^{j_\iota} \times T^{-i_\iota}$ can have nonempty intersection with those $A_{i_\kappa}^{j_\kappa} \times T^{-i_\kappa}$ such that $i_\kappa \neq i_\iota$. By definition of $\mathbf{q}^{\mathscr{A}}$, therefore,

$$q_{i_\iota}^{\mathscr{A}}|_{A_{i_\iota}^{j_\iota} \times T^{-i_\iota}} \equiv 0,$$

which yields that $Q^{\mathscr{A}}_{i_\iota}(j_\iota) = 0$ for all $\iota > K$.

Since for all $\iota = K+1, \ldots, L$, we have $\alpha^{j_\iota}_{i_\iota} \leq 0$, $\alpha^{j_K}_{i_K} > 0$, $\bar{P}_{i_\iota}(j_\iota) \geq 0$ and $Q^{\mathscr{A}}_{i_\iota}(j_\iota) = 0$, it follows from (36),

$$0 > \sum_{\iota=1}^{K} \left(Q^{\mathscr{A}}_{i_\iota}(j_\iota) - \bar{P}_{i_\iota}(j_\iota) \right). \tag{37}$$

Namely, we show that $\langle \chi_{A(K)}, \bar{\mathbf{P}} - \bar{\mathbf{Q}}^{\mathscr{A}} \rangle > 0$, or equivalently by the Claim (30),

$$\langle \chi_{A(K)}, \bar{\mathbf{P}} \rangle > \mu \left(\bigcup_{i=1}^{N} A_i(K) \times T^{-i} \right),$$

which completes our proof. ∎

3 Conclusion

In this note, I generalized the results in [3] to *continuous* types. As a future direction, we wish to show, in the same spirit of [2], that the above theorem can be simplified by using a much smaller test subsets instead of all the measurable subsets of $\mathcal{T}$. That is the following corollary:

Corollary 9 *Let* $\mathbf{P} = (P_1, P_2, \cdots, P_N)$ *and* $E^\alpha_i := \{\tau \in T_i : P_i(\tau) \geq \alpha\}$ *for any* $\alpha \in [0,1]$, $i = 1, 2, \ldots, N$. *And define* $E^a := \amalg_{i=1}^{N} E^{\alpha_i}_i$ *for any given vector* $a = (\alpha_1, \alpha_2, \cdots, \alpha_N) \in [0,1]^N$. *Then if for any given* $a \in [0,1]^N$, $\mathbf{P}$ *satisfies*

$$\langle \chi_{E^a}, \mathbf{P} \rangle \leq \mu(\{\mathbf{t} \in T : \exists i, \quad s.t. \quad t_i \in E^{\alpha_i}_i\}), \tag{38}$$

then $\mathbf{P}$ *satisfies* (19) *for any measurable* $A \subset \mathcal{T}$.

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