Technical Note

A Simple Application of Seminorm Representation in Dynamic Programming

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1 Introduction

In this technical note, we demonstrate the dual space of $\mathbb{R}^{\Theta}$ (the space of (continuous) functions on a discrete topological space Θ) (Proposition 7 in this note), which is essentially the representation of C3/C5 properties in [3], where they utilize a characterization of discrete topology from [2].

In this note, I will prove this representation result from a more abstract point of view, in the hope that this approach may be applied to other topology structure in the future. The key idea is to recognize that the representation of C3/C5 properties in [3] is actually a special case of the linear functional representation in a locally convex TVS once we identify the "right" seminorm representation of the discrete topology.

Section 2 introduces the concepts and results that we are going to use in the note. Section 3 proves the representation of C3/C5 properties in [3] by embedding the space of $\mathbb{R}^{\Theta}$ in a locally convex TVS equipped with with suitable seminorm. I finally point out some potential applications of this abstract approach in Section 4.

2 Locally Convex TVS and Riesz Representation Theorem

Just in order to be self-contained, I first hereby give the basic concepts and already well-known results from topological vector spaces. All the following notation and results follow from [1][1].

[1]Different textbooks may have different definitions and notations, but I will use the version stated in [1].

Definition 1 *A (real)* topological vector space *(TVS) is a vector space $\mathscr{X}$ together with a topology, w.r.t which:*

1. *the vector addition $\mathscr{X} \times \mathscr{X} \to \mathscr{X}$ defined by $(x, y) \to x + y$ is continuous;*
2. *the scaler multiplication $\mathbb{R} \times \mathscr{X} \to \mathscr{X}$ defined by $(\lambda, x) \to \lambda x$ is continuous.*

Definition 2 *A* seminorm *on a TVS $\mathscr{X}$ is a function $p : \mathscr{X} \to [0, +\infty)$ with the following properties:*

1. subadditivity*: $\forall x, y \in \mathscr{X}, p(x + y) \leq p(x) + p(y)$;*
2. homogeneity*: $\forall \lambda \in \mathbb{R}, x \in \mathscr{X} : p(\lambda x) = |\lambda| p(x)$;*

Definition 3 *A* locally convex *TVS $\mathscr{X}$ is a TVS whose topology is defined by a family of seminorms $\mathscr{P}$ such that*

1. *$\cap_{p \in \mathscr{P}} \{x \in \mathscr{X} : p(x) = 0\} = \{0\}$;*
2. *a subset U of $\mathscr{X}$ is* open *iff $\forall x_0 \in \mathscr{X}$ there are $p_1, \cdots, p_n$ in $\mathscr{P}$ and $\epsilon_1, \cdots, \epsilon_n > 0$ such that $\cap_{j=1}^{n} \{x \in \mathscr{X} : p_j(x - x_0) < \epsilon_j\} \subset U$.*

The following proposition provides a characterization of continuous linear functionals on a locally convex TVS. The finiteness of this characterization in proposition will play a fundamental role in the proof of Section 3.

Proposition 4 *$F : \mathscr{X} \to \mathbb{R}$ is a continuous linear functional on a locally convex TVS defined by seminorms $\mathscr{P}$ if and only if there exist* finitely many *$p_1, p_2, \ldots, p_l$ in $\mathscr{P}$ and positive scalers $\lambda_1, \lambda_2, \ldots, \lambda_l$ such that*

$$|F(x)| \leq \sum_{i=1}^{l} \lambda_i p_i(x), \forall x \in \mathscr{X}.$$

Here we quote the famous Riesz Representation Theorem as a reference for possible future generalization. But in this note, we will only use a very limited version of the Theorem (see Remark 1), which is basically the linear functional representation on a finite dimensional space and can actually be derived from elementary linear algebra.

Theorem 5 (Riesz Representation) *Let V be a locally compact space, $M(V)$ the space of regular Borel signed measures and $C_0(V)$ the space of continuous functions on V which vanish at infinity. For each $\mu \in M(V)$, define $F_\mu : C_0(V) \to \mathbb{R}$ by*

$$F_\mu(f) := \int f d\mu, \forall f \in C_0(V).$$

Then $F_\mu \in C_0(V)^*$ and the map $\mu \to F_\mu$ is an isometric isomorphism of $M(V)$ onto $C_0(V)^*$.

Remark 1 *We will later use a special case of this theorem where V will be a finite set $\{v_1, v_2, \ldots, v_l\}$ with discrete topology (and hence is automatically locally compact). In that case, $M(V)$ is just a space of l-dimensional real vector $(\mu_1, \mu_2, \ldots, \mu_l)$ with number $\mu_i \in \mathbb{R}$ assigned to point v_i for $i = 1, 2, \ldots, l$. And $C_0(V) = \mathbb{R}^V$ by virtue of the fact that any function on a discrete topology is continuous and V is finite.*

3 Proof of Key Result in [3]

Let Θ be a given topological space with *discrete topology* and denote as $\mathbb{R}^\Theta$ the space of all real-valued functions[2] defined on Θ with the *pointwise convergence topology.* [3] has already pointed out that the $\mathbb{R}^\Theta$ is a locally convex TVS equipped with the natural product topology inherited from $\mathbb{R}$. Here we pointed out the topology of $\mathbb{R}^\Theta$ can be re-defined by the following family of seminorms:

Lemma 6 *Let $\mathscr{K}$ be the collection of finite subsets (hence they are compact) of Θ, i.e. $\mathscr{K} := \{K \subset \Theta : 0 < |K| < \infty\}$, where $|K|$ denotes the cardinality of subset K. Define seminorm*[3]

$$p_K(f) := \sup\{|f(\theta)| : \theta \in K\}, \forall f \in \mathbb{R}^\Theta \quad (1)$$

and $\mathscr{P} := \{p_K : K \in \mathscr{K}\}$. Then the product topology of $\mathbb{R}^\Theta$ is given by $\mathscr{P}$.

Proof: First, we verify that such a $\mathscr{P}$ is well-defined. Notice that the singletons belong to $\mathscr{K}$, and hence if $p(f) = 0$ for all $p \in \mathscr{P}$, then we have $f(\theta) = 0$ for all $\theta \in \Theta$ and hence $f \equiv 0$.

Notice that the basis for the product topology $\mathbb{R}^\Theta$ is of the form $\prod_{\theta\in\Theta} O_\theta$ for O_θ open in $\mathbb{R}$ and $O_\theta = \mathbb{R}$ except for finitely many θ (see [2]). While the basis for the locally convex space $\mathbb{R}^\Theta$ is of the form $\prod_{i=1}^m \{f \in \mathbb{R}^\Theta : p_{K_i}(f) < \epsilon_i\}$ with m finite (see [1]). Due to the finiteness of K_i, it is immediate to see that these two topologies are equivalent. ■

[2]Since Θ is equipped with discrete topology, $\mathbb{R}^\Theta = C(\Theta)$ the space of continuous functions on Θ.
[3]Actually, we may write $p_K(f) := \max\{|f(\theta)| : \theta \in K\}$, since K is finite.

Remark 2 *If $K \subset K'$ for $K, K' \in \mathscr{K}$, then we must have $p_K \leq p_{K'}$.*

The following result is the key of this note. It provides the characterization/representation of *hyperplanes* in $\mathbb{R}^\Theta$.

Proposition 7 *$F : \mathbb{R}^\Theta \to \mathbb{R}$ is a continuous linear functional on $\mathbb{R}^\Theta$, i.e. $F \in (\mathbb{R}^\Theta)^*$,* if and only if *there exist finitely many $\theta_1, \theta_2, \ldots, \theta_l$ and signed measure on them $\mu_1, \mu_2, \ldots, \mu_l$ such that for all $f \in \mathbb{R}^\Theta$*

$$F(f) = \sum_{i=1}^{l} \mu_i f(\theta_i). \tag{2}$$

Proof: The "If" direction is obvious, so we just concentrate on the "only if" part. Suppose $F \in (\mathbb{R}^\Theta)^*$. Then by Proposition 4 and Lemma 6, there are $K_1, K_2, \ldots, K_n \in \mathscr{K}$ (n is finite) and positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $|F(f)| \leq \sum_{j=1}^{n} \lambda_j p_{K_j}(f)$. Let $K := \cup_{j=1}^{n} K_j \in \mathscr{K}$ and $\lambda = \max\{\lambda_j : 1 \leq j \leq n\}$. Then $|F(f)| \leq \lambda p_K(f)$ by Remark 2. Hence, if $f \in \mathbb{R}^\Theta$ and $f|K \equiv 0$, then $F(f) = 0$.

Define a linear functional $L : \mathbb{R}^K \to \mathbb{R}$ as follows: if $g \in \mathbb{R}^K := C_0(K)$ by Remark 1, let $\tilde{g} \in \mathbb{R}^\Theta := C(\Theta)$ be such that $\tilde{g}|K = g$, i.e. $\tilde{g}$ is a continuous extension of g, and define $L(g) := F(\tilde{g})$. It is very easy to verify that L is well-defined and is indeed linear. Now notice that since K is finite, the seminorm $p_K(\cdot)$ is actually the sup-norm $\|\cdot\|_{C_0(K)}$ on $C_0(K)$, hence we have

$$|L(g)| = |F(g)| \leq \lambda p_K(\tilde{g}) = \lambda \|g\|_{C_0(K)}.$$

Therefore, by Theorem 5 and its Remark 1, there exists a signed measure $\mu \in M(K)$ such that $L(g) = \int_K g d\mu = \sum_{j=1}^{|K|} \mu_j g(\theta_j)$ where θ_j is the generic element of K, since K is finite. Hence, for any $f \in \mathbb{R}^\Theta$, we have

$$F(f) = L(f|K) = \int_K f d\mu = \sum_{j=1}^{|K|} \mu_j f(\theta_j).$$

■

With the preparation of this key proposition, using the same hyperplane arguments as [3] did: *any closed convex subset in a locally convex space can be written as an intersection of a family of closed affine hyperplanes*, we then can easily obtain the key result in [3]:

Theorem 8 (Proposition 1 in [3]) *A property P is a C3 property if and only if there exists a collection, indexed by $\alpha \in \mathscr{A}$, of finite sets of points $\{\theta_\beta\}_{\beta \in B_\alpha}$, $\{\theta_\gamma\}_{\gamma \in \Gamma_\alpha}$ and positive weights $\{\lambda_\beta\}_{\beta \in B_\alpha}$, $\{\lambda_\gamma\}_{\gamma \in \Gamma_\alpha}$ that define a test of satisfaction of the form: f satisfies P if and only if*

$$\sum_{\beta \in B_\alpha} \lambda_\beta f(\theta_\beta) \leq \sum_{\gamma \in \Gamma_\alpha} \lambda_\gamma f(\theta_\gamma).$$

Furthermore, if P is a C5 property, for each $\alpha in \mathscr{A}$, we can normalize the weights to sum to one.

4 Conclusion

From [3], we know that the core of the proof of Theorem 8 is to prove Proposition 7 here in this note. My approach is to represent pointwise-convergence topology as a locally convex TVS equipped with an appropriate seminorm, which then allows me to apply the linear functional representation given in Proposition 4. This approach, together with the Riesz Representation Theorem, may provide other useful linear functional representations under different topologies.

References

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